A note on the generalized q-Euler numbers(2)

By

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Abstract. Recently the new q-Euler numbers and polynomials related to Frobenius-Euler numbers and polynomials are constructed by Kim (see[3]). In this paper, we study the generalized q-Euler numbers and polynomials attached to χ related to the new q-Euler numbers and polynomials which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized q-Euler numbers and polynomials attached to χ.

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§1. Introduction

Let $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of real numbers and the complex number field. and let p be a fixed an odd prime number.Assume that q is an indeterminate in $\mathbb{C}$ with $q \in \mathbb{C}$ with $|q| < 1$. As the q-symbol $[x]_q$, we denote $[x]_q = \frac{1-q^x}{1-q}$. Recently, q-Euler polynomials are defined as

$$\frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \text{ for } |t+\log q| < \pi, \text{ (see [3])}.$$

In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ are call the n-th q-Euler numbers (see [3]). These q-Euler numbers and polynomials are closely relayed to Frobenius-Euler numbers and polynomials and these numbers are studied by Simsek-Cangul-Ozden, Cenkci-Kurt and Can and several authors (see [1-2, 18-26]). In this paper, we study the generalized q-Euler numbers and polynomials attached to χ related to the q-Euler numbers and polynomials, $E_{n,q}(x)$, which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized q-Euler numbers and polynomials attached to χ.

§2. Congruence for q-Euler numbers and polynomials

The ordinary Euler polynomials are defined as

$$e^{xt}\frac{2}{e^t+1}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad \text{(see [1-5])},$$

where we use the technical method notation by replacing $E^n(x)$ by $E_n(x)(n\geq 0)$, symbolically (see [1-2]). Let us consider the generating function of q-Euler polynomials $E_{n,q}(x)$ as follows:

$$F_q(x,t)=\frac{[2]_q}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!},\tag{1}$$

and we also note that

$$\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=\frac{[2]_q}{qe^t+1}e^{xt}=\frac{1-(-q^{-1})}{e^t-(-q^{-1})}=\sum_{n=0}^{\infty}H_n(-q^{-1},x)\frac{t^n}{n!},$$

where $H_n(-q^{-1},x)$ are called the n-th Frobenius-Euler polynomials (see [3]). From (1), we note that

$$\lim_{q\to 1}F_q(x,t)=\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.\tag{2}$$

By (1) and (2), we see that

$$\lim_{q\to 1}E_{n,q}(x)=E_n(x).$$

In (1), it is easy to show that

$$\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=F_q(x,t)=\frac{[2]_q}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l,q}x^{n-l}\right)\frac{t^n}{n!}.$$

By comparing the coefficients on the both sides, we have

$$E_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}x^{n-l},\quad \text{where } E_{l,q} \text{ are the } l\text{-th } q\text{-Euler numbers.}\tag{3}$$

Let χ be the Dirichlet's character with conductor $d\equiv 1 \pmod 2$. Then we define generating function of the generalized q-Euler numbers attached to χ, $E_{n,\chi,q}$ as follows:

$$F_{q,\chi}(t)=\frac{[2]_q\sum_{l=0}^{d-1}\chi(l)q^l(-1)^le^{lt}}{q^de^{dt}+1}=\sum_{n=0}^{\infty}E_{n,\chi,q}\frac{t^n}{n!}.\tag{4}$$

From (4), we note that

$$\lim_{q\to 1}F_{q,\chi}(t)=\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^ae^{at}}{e^{dt}+1}=\sum_{n=0}^{\infty}E_{n,\chi}\frac{t^n}{n!},\tag{5}$$

where $E_{n,\chi}$ are the n-th ordinary Euler numbers attached to χ. By (4) and (5), we see that

$$\lim_{q\to 1} E_{n,\chi,q} = E_{n,\chi}.$$

From (5), we can also derive

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!} &= F_{q,\chi}(t) = [2]_q \sum_{k=0}^{\infty}\chi(k)(-q)^k e^{kt} \\ &= \sum_{n=0}^{\infty}\left([2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k k^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{a=0}^{d-1}(-q)^a\chi(a)E_{n,q^d}(\frac{a}{d})\right)\frac{t^n}{n!}.\end{aligned} \tag{6}$$

By comparing the coefficients on the both sides of (6), we have

$$E_{n,\chi,q} = [2]_q \sum_{k=0}^{\infty}\chi(k)(-q)^k k^n = d^n\sum_{a=0}^{d-1}(-q)^a\chi(a)E_{n,q^d}(\frac{a}{d}). \tag{7}$$

Finally, we define the generating function of the generalized q-Euler polynomials attached to χ, $E_{n,\chi,q}(x)$ as follows:

$$F_{q,\chi}(x,t) = \sum_{n=0}^{\infty}E_{n,\chi,q}(x)\frac{t^n}{n!} = [2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k e^{(x+k)t}. \tag{8}$$

By (8), we easily see that

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} &= F_{q,\chi}(x,t) = [2]_q \sum_{k=0}^{\infty}\chi(k)(-q)^k e^{(x+k)t} \\ &= \sum_{n=0}^{\infty}\left([2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k (x+k)^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{a=0}^{d-1}(-q)^a\chi(a)E_{n,q^d}(\frac{a+x}{d})\right)\frac{t^n}{n!}.\end{aligned} \tag{9}$$

Thus, we have

$$E_{n,\chi,q}(x) = d^n\sum_{a=0}^{d-1}(-q)^a\chi(a)E_{n,q^d}(\frac{a+x}{d}) = \sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}E_{\ell,\chi,q} = [2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k(x+k)^n. \tag{10}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then, we see that

$$\begin{aligned}q^d F_{q,\chi}(d,t) + F_{q,\chi}(t) &= [2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k e^{(d+k)t} + [2]_q\sum_{k=0}^{\infty}\chi(k)(-q)^k e^{kt} \\ &= [2]_q\sum_{k=0}^{d-1}\chi(k)(-q)^k e^{kt}.\end{aligned} \tag{11}$$

From (11), we have

$$\sum_{n=0}^{\infty}\left(q^d E_{n,\chi,q}(d)+E_{n,\chi,q}\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{[2]_q\sum_{k=0}^{d-1}\chi(k)(-q)^k k^n\right\}\frac{t^n}{n!}.$$

Therefore, we obtain the following theorem.

THEOREM 1. *For $q\in\mathbb{C}$ with $|q|<1, n\in\mathbb{Z}_+$ and $d\in\mathbb{N}$ with $d\equiv 1\,(\mathrm{mod}\ 2)$, we have*

$$q^d E_{n,\chi,q}(d)+E_{n,\chi,q}=[2]_q\sum_{k=0}^{d-1}\chi(k)(-q)^k k^n.$$

Let p be a positive odd integer and let $N\in\mathbb{N}$. Then we have

$$\begin{aligned}[2]_q\sum_{a=0}^{dp^N-1}\chi(a)(-q)^a a^n &= q^{dp^N}E_{n,\chi,q}(dp^N)+E_{n,\chi,q}\\ &= q^{dp^N}\sum_{j=0}^{n}\binom{n}{j}(dp^N)^j E_{n-j,\chi,q}+E_{n,\chi,q}\\ &= q^{dp^N}\sum_{j=1}^{n}\binom{n}{j}(dp^N)^j E_{n-j,\chi,q}+(q^{dp^N}+1)E_{n,\chi,q}\\ &\equiv 2E_{n,\chi,q}\,(\mathrm{mod}\ dp^N),\end{aligned}$$

because $q^{ndp^N}\equiv 1\,(\mathrm{mod}\ dp^N)$. Therefore, we obtain the following theorem.

THEOREM 2. *Let p be a positive odd integer and $q\in\mathbb{C}$ with $|q|<1$ and $(q-1,dp)=1$. For $d\in\mathbb{N}$ with $d\equiv 1\,(\mathrm{mod}\ 2)$, we have*

$$[2]_q\sum_{a=0}^{dp^N-1}\chi(a)(-q)^a a^n\equiv 2E_{n,\chi,q}\,(\mathrm{mod}\ dp^N).$$

REMARK. *Define*

$$L_{E,q}(s,\chi|x)=[2]_q\sum_{n=0}^{\infty}\frac{(-q)^n\chi(n)}{(n+x)^s},$$

where $s\in\mathbb{C}$, and $x\neq 0,-1,-2,\cdots$. For $k\in\mathbb{Z}_+$, we have $L_{E,q}(-k,\chi|x)=E_{k,\chi,q}(x)$.

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REFERENCES

[1] R. P. Agarwal, C. S. Ryoo, *Numerical Computations of the roots of the generalized twisted q-Bernoulli polynomials*, Neural parallel Sci. Comput, **15** (2007), 193-206.

[2] I. N. Cangul, H. Odzen and Y. Simsek, *Generating Functions of the (h, q)-extension of Euler polynomials and numbers*, Acta Math. Hungar. **120** (2008), 281-299.

[3] T. Kim, *An invariant p-adic q-integral on $\mathbb{Z}_p$*, Appl. Math. Letters **21** (2008), 105-108.

[4] T. Kim, *q-extension of the Euler formula and trigonometric functions*, Russ. J. Math. Phys. **14 (3)** (2007),275-278.

[5] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl. **14** (2008), 1267-1277.

[6] H. M. Srivastava, T. Kim, and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12(2)** (2005), 241-268.

[7] T. Kim, *Note on q-Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. **17** (2008), 9-15.

[8] T. Kim, *The modified q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **16** (2008), 161-170.

[9] T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory, **76** (1999), 320-329.

[10] T. Kim, *q-Volkenborn Integration*, Russ. J. Math. Phys. **9** (2002), 288-299.

[11] T. Kim, *q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients*, Russ. J. Math. Phys. **15** (2008), 51-57.

[12] T. Kim, J. Y. Choi, and J. Y. Sug, *Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **14** (2007), 160-163.

[13] T. Kim, *On the von Staudt-Clausen's Theorem for the q-Euler numbers*, Russ. J. Math. Phys. **16 (3)** (2009).

[14] T. Kim, *q-generalized Euler numbers and polynomials*, Russ. J. Math. Phys.**13(3)**(2006), 293-298.

[15] T. Kim, *Multiple p-adic L-function* Russ. J. Math. Phys. **13(2)** (2006), 151-157.

[16] T. Kim, *Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function*, Russ. J. Math. Phys. **12(2)** (2005), 186-196.

[17] T. Kim, *Analytic continuation of multiple q-zeta functions and their values at negative integers*, Russ. J. Math. Phys. **11(1)** (2004), 71-76.

[18] T. Kim *On Euler-Barnes multiple zeta functions*, Russ.J. Math. Phys. **10(3)** (2003), 261-267.

[19] B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials* , J. Nonlinear Math. Phys. **12** (2005), 412-422.

[20] H. Ozden, Y. Simsek, *Interpolation functions of the* (h, q)*-extension of twisted Euler numbers*, Comput. Math. Appl. **56** (2008), 898-908.

[21] H. Ozden, Y. Simsek, *A new extension of q-Euler numbers and polynomials related to their interpolation functions*, Appl. Math. Lett. **21** (2008), 934-939.

[22] M. Schork, *Ward's calculus sequence; q-calculus and their limit* $q \to -1$, Adv. Stud. Contemp. Math. **13** (2006), 131-141.

[23] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math. **16** (2008), 251-278.

[24] Y. Simsek, *p-adic twisted q-L-functions related to generalized twisted Bernoulli numbers*, Russ. J. Math. Phys. **13(3)** (2006), 340-348.

[25] Y. Simsek, O. Yurekli, and V. Kurt, *On interpolation functions of the twisted generalized Frobenius-Euler numbers*, Adv. Stud. Contemp. Math. **15(2)** (2007),187-194.

[26] S. H. Rim, K. H. Park, and E.J. Moon, *On Genocchi numbers and polynomials*, Abstract and Applied Analysis, Article ID **898471**, (2008) 7 pages, dio:10.1155/2008/898471.

[27] T. Kim, Y.H. Kim, K. W. Hwang *On the q-extensions of the Bernoulli and Euler numbers, related identities and lerch zeta function*, Proc. Jangjeon Math. Soc. **12** (2009), no. 1, 1798–1804.

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